# Ion motion and finite temperature effect on relativistic strong plasma waves

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The influence of motion of ions and electron temperature on nonlinear one-dimensional plasma waves with velocity close to the speed of light in vacuum is investigated. It is shown that although the wave-breaking field weakly depends on the mass of ions, the nonlinear relativistic wavelength essentially changes. The nonlinearity leads to the increase of the strong plasma wavelength, while the motion of ions leads to the decrease of the wavelength. Both the hydrodynamic approach and kinetic one, based on Vlasov-Poisson equations, are used to investigate the relativistic strong plasma waves in a warm plasma. The existence of relativistic solitons in a thermal plasma is predicted. [S1063-651X(98)12611-9]

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## I. INTRODUCTION

Strong plasma waves passing through a plasma with phase velocity slightly smaller than the velocity of light have been the subject of much interest during the past two decades. Such waves can be excited in plasma by relativistic bunches of charged particles or laser pulses. The excited plasma waves can be used both to accelerate charged particles and to focus charged bunches [1]. Plasma-based accelerator concepts are currently under intensive development (see overview in Ref. [2] and numerous references therein). Accelerating gradients in the plasma wave can reach the value of tens of GeV/m (notice that in conventional linacs accelerating gradients are on the order of tens of MeV/m) that is confirmed in recent experiments [2]. Also, the value of the focusing field can be much greater than that reached in conventional focusing magnetic systems. The acceleration of charged particles by relativistic strong waves also is under consideration as a possible mechanism of ultrahigh energy (up to $10^{20}$ eV) cosmic ray generation in astrophysical plasma.

In a cold plasma the amplitude of a one-dimensional plasma wave is limited by the wave-breaking field. In the nonrelativistic case, when the wave phase velocity $v_{ph}$ is much less than the velocity of light ($v_{ph} \ll c$), the wave-breaking amplitude is equal to [3] $E_* = m_e \omega_{pe} v_{ph}/|e|$, where $\omega_{pe} = (4\pi n_0 e^2/m_e)^{1/2}$ is the electron plasma frequency, $n_0$ is the density of electrons in unperturbed plasma, $m_e$ and $e$ are the electron rest mass and their charge; the ions assumed to be immobile. In the relativistic case the wave-breaking field is equal to [4] $E_{rel} = [2(\gamma-1)]^{1/2}/\beta$; here $\beta = v_{ph}/c$, $\gamma = (1-\beta^2)^{-1/2}$ is the relativistic factor and $E_{rel}$ is normalized on $E_*$. The one-dimensional relativistic strong waves (RSW's) can be excited in a plasma by wide relativistic bunches of charged particles or intensive laser pulses [2] (when $k_p a \gg 1$, where $k_p = \omega_{pe}/v_{ph}$, $a$ is the characteristic transverse sizes of bunches or pulses).

Another important characteristic of nonlinear plasma waves is the dependence of the wavelength on the wave amplitude. Both in the linear case and in the nonlinear nonrelativistic one, the plasma wavelength in cold plasma is $\lambda_p = 2\pi v_{ph}/\omega_{pe}$. In the relativistic nonlinear regime, when plasma electrons get a relativistic velocity in the process of oscillations, nonlinear wavelength increases with the amplitude [5–8]. In the ultrarelativistic case ($\gamma \gg 1$), for the wave amplitude $E_{mp} \cong E_{rel}$, the wavelength is approximately equal to [5] $4(2\gamma)^{1/2}\lambda_p$. One can see that for large $\gamma$ the wavelength is essentially more than the usual linear plasma wavelength.

In the previous studies the plasma ions, in the process of oscillations, including the nonlinear relativistic waves, were usually assumed to be immobile due to their large mass. In Ref. [9] it is shown that when $\gamma \ll (M/16m_e)^{1/3}$ (here $M$ is the mass of an ion; for example, for hydrogen plasma, consisting of protons and electrons, this condition gives $\gamma < 5$), the wave-breaking amplitude is approximately equal to $E_{rel}$ and the motion of ions can be neglected. However, the dispersion properties of the relativistic strong plasma waves, which take into consideration the motion of ions, have not been elucidated up to now. This problem has been considered in Sec. II on the basis of cold hydrodynamics equations for the arbitrary $\gamma$ and mass of particles forming the plasma. The necessity of taking into account the ion motion is conditioned by the following reasons. First, because the maximum relativistic wavelength and amplitude grow in proportion to $\gamma^{1/2}$, the plasma ions (even heavy ions) in such a strong field can reach a velocity that is sufficient to make an essential contribution in the process of charge separation in the wave. On the other hand, in a semiconductor plasma positively charged particles (holes) have a mass similar to or less than that of an electron. The problem also has an astrophysical aspect. The pole region of the pulsars is considered to be filled with an electron-positron plasma in which the strong plasma waves can be excited and high-energy charged particles generated [10]. It is obvious that in the plasma wave passing through an electron-positron plasma, neither electrons nor positrons may be considered as a neutralizing background.

Another important problem is the influence of plasma temperature on RSW's. Finite plasma temperature has decisive significance for a description of RSW's near wave-breaking. Actually, according to the one-dimensional (1D) theory of relativistic plasma waves in a cold plasma, at the breaking point hydrodynamic plasma electron velocity is equal to the phase velocity and the density of electrons $n_e$ tends to infinity [5–7,11] (in this case the spatial behavior of the density is similar to the $\delta$ function). On the other hand, in a thermal plasma (even when the temperature is low), the pressure tends to infinity when $n_e \to \infty$. Thus, in this case, the pressure, as well as the plasma temperature, should be taken into consideration. In Ref. [12] the finite plasma temperature effect on the nonrelativistic ($v_{ph} \ll c$) wave-breaking field is considered using a 1D waterbag model for the distribution function. In this model it is assumed that the electron distribution function during oscillations is constant in a limited interval of velocities and is equal to zero outside of this interval. It is shown [12] that the maximum amplitude of the plasma waves decreases with temperature. In Ref. [13] the 1D relativistic waterbag model is used to investigate RSW's in a warm plasma. Using the relativistic equation of motion with the pressure term for plasma electrons, in Ref. [11] it is shown that in the case $\gamma \gg m_e v_{ph}^2/3T$ (where $T$ is the temperature of electrons) the wave-breaking field is proportional to $T^{-1/4}$. The authors of Ref. [14] have analyzed the influence of low temperature ($T \ll m_e c^2$) on excitation of nonlinear wake fields by relativistic charged bunches. They considered the equation of motion obtained using second moments of the distribution function. In the present paper (Sec. III) the hydrodynamics equations are used to study dispersion properties of RSW's in a warm plasma. The dispersion correlation for the weakly nonlinear case is obtained. In Sec. IV the strong plasma waves are investigated on the basis of the relativistic Vlasov kinetic equation and the Poisson equation.

## II. ION MOTION EFFECT ON DISPERSION PROPERTIES OF RELATIVISTIC STRONG PLASMA WAVES

In this section we consider a cold uniform plasma consisting of positively charged particles (for example, protons or positrons) with mass $m_+$ and electric charge $q_+$, and negatively charged particles (electrons or negatively charged ions) with mass $m_-$ and charge $q_-$. The relativistic equation of motion, the continuity equation for each plasma component, and the Poisson equation for one-dimensional steady plasma waves are

$$(\beta-\beta_{\pm})\frac{d(\beta_{\pm}\gamma_{\pm})}{dz} = -\frac{q_{\pm}}{|q_-|}\beta^2 E, \tag{1}$$

$$\beta\frac{dN_{\pm}}{dz} - \frac{d(N_{\pm}\beta_{\pm})}{dz} = 0, \tag{2}$$

$$\frac{dE}{dz} = 1 - N_- + |q_+/q_-|N_+, \tag{3}$$

where $z = k_p(Z - v_{ph}t)$, $k_p = \omega_p/v_{ph}$, $\omega_p = (4\pi n_{0-}q_-^2/m_-)^{1/2}$, $n_{0-}$ is the density of negatively charged particles in equilibrium, $\beta_{\pm} = v_{\pm}/c$ are dimensionless velocities, $\gamma_{\pm} = (1-\beta_{\pm}^2)^{-1/2}$, and densities $N_{\pm}$ are normalized on the equilibrium values. The electric field strength is normalized on the nonrelativistic wave-breaking field $m_-\omega_p v_{ph}/|q_-|$ and obeys the formula

$$E(z) = -(1/\beta^2)d\Phi/dz, \tag{4}$$

where $\Phi \equiv \Phi_- = 1 + |q_-|\varphi/m_-c^2 \geq 1/\gamma$, $\varphi$ is the electric potential. From expressions (1), (2) and (4) we have

$$\beta_{\pm} = [\beta - (\Phi_{\pm}^2 - \gamma^{-2})^{1/2}]/(\beta^2 + \Phi_{\pm}^2), \tag{5}$$

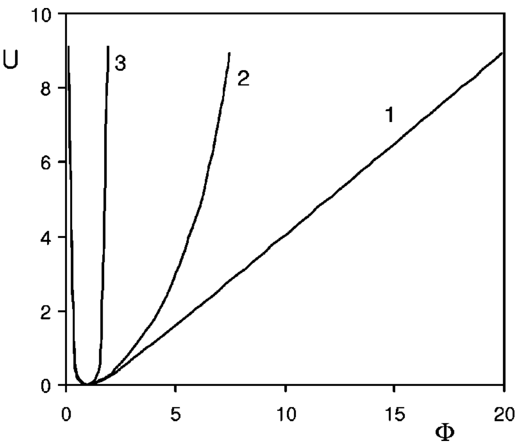


FIG. 1. "Potential" $U(\Phi)$ for the value $\gamma = 10$ (in dimensionless units). $1-\mu=0$; $2-\mu=0.1$; $3-\mu=1$.

$$N_{\pm} = \beta\gamma^2[\Phi_{\pm}/(\Phi_{\pm}^2 - \gamma^{-2})^{1/2} - \beta]. \tag{6}$$

Substituting $N_{\pm}(\Phi_{\pm})$ and expression (4) in the Poisson equation (3) one obtains the following differential equation of the second order for $\Phi$:

$$\frac{d^2\Phi}{dz^2} + \beta^3\gamma^2\left(\frac{\Phi_+}{(\Phi_+^2-\gamma^{-2})^{1/2}} - \frac{\Phi}{(\Phi^2-\gamma^{-2})^{1/2}}\right) = 0, \tag{7}$$

Here $\Phi_+ = 1 - q_+\varphi/m_+c^2 = 1 + \mu(1-\Phi) \geq 1/\gamma$ and $\mu = |q_+/q_-|m_-/m_+$. The electric potential $\varphi$ is assumed to be equal to zero when the plasma density is equal to the equilibrium density.

Equation (7) can be rewritten in the form

$$\frac{d^2\Phi}{dz^2} + \frac{dU}{d\Phi} = 0,$$
$$U = \beta^3\gamma^2\{[\beta - (\Phi_+^2-\gamma^{-2})^{1/2}]/\mu + [\beta - (\Phi^2-\gamma^{-2})^{1/2}]\}. \tag{8}$$

Here, for convenience, $U(\Phi)$ is chosen to be equal to zero at a point $\Phi = 1$, where it reaches a minimum. When $\mu \to 0$, Eq. (7) reduces to the known equation for nonlinear waves in a plasma with immobile ions [2,7]. Formally, Eq. (8) describes the one-dimensional motion of a particle in a field with potential $U(\Phi)$; the values $\Phi$ and $E$ correspond to the coordinate and velocity of this fictitious particle, respectively. Function $U$ determines the characteristic of the field moving through the plasma. In Fig. 1 this function is presented for $\gamma = 10$ ($\beta \approx 0.995$) and different values of $\mu$. One can see that for the arbitrary parameters the solutions of Eq. (8) [or Eq. (7)] are the periodic plasma waves (including the wave with zero amplitude-unperturbed plasma). Integrating Eq. (8) we have

$$\frac{d\Phi}{dz} = -\beta^2 E = \pm[2(U_{\max} - U)]^{1/2}, \tag{9}$$

where $U_{\max}$ is maximum value of $U(\Phi)$ in the process of oscillations. From Eq. (9) it follows that the plasma wave amplitude is equal to $E_{mp} = (2U_{\max})^{1/2}/\beta^2$. Substituting the maximum permissible value of $U(\Phi)$, reaching the point $\Phi = 1/\gamma$, in this expression, we find the wave-breaking field,

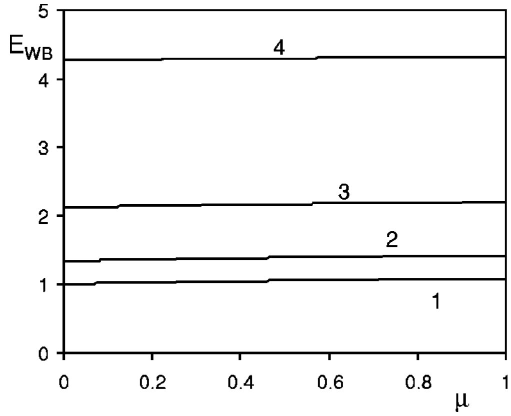


FIG. 2. Normalized wave-breaking field depending on $\mu$. $1-\gamma=1.01$; $2-\gamma=1.5$; $3-\gamma=3$; $4-\gamma=10$.

$$E_{WB}=2^{1/2}\gamma[1+(1-\xi_1^{1/2}\xi_2^{1/2})/\mu],$$
$$\xi_1=1+\mu, \quad \xi_2=1+\mu(\gamma-1)/(\gamma+1). \tag{10}$$

In the case $\mu\ll 1$, from Eq. (10) follows the expression

$$E_{WB}\approx(1+\mu/8)[2(\gamma-1)]^{1/2}/\beta, \tag{11}$$

which reduces to the well known relativistic wave-breaking field for plasma with immobile ions, when $\mu=0$ [4]. Figure 2 shows the wave-breaking field $E_{WB}$ depending on $\mu$ (for example, for electron-positron plasma $\mu=m_e/m_{pos}=1$, for the hydrogen plasma $\mu=m_e/m_{prot}\approx 5.455\times 10^{-4}$) for different values of $\gamma$. In both the nonrelativistic case and relativistic one the wave-breaking field weakly increases with $\mu$. For example, according to Eq. (10), in the nonrelativistic case ($\gamma\approx 1$), $E_{WB}(\mu=1)$ only $2(1-2^{-1/2})^{1/2}\approx 1.08$ times exceeds the wave-breaking amplitude at $\mu=0$. Proceeding from the shape of the ''potential'' $U(\Phi)$ (see Fig. 1) one can expect that the plasma wavelength undergoes considerable change with $\mu$. In Fig. 3 the dependence of the relativistic plasma wavelength $\Lambda_p$ [note that, according to the variables accepted in Eqs. (1)–(3), the linear plasma wave at $\mu=0$ corresponds to the value $\Lambda_p=2\pi$] on the amplitude presented. Curve 1 corresponds to the case of immobile ions and

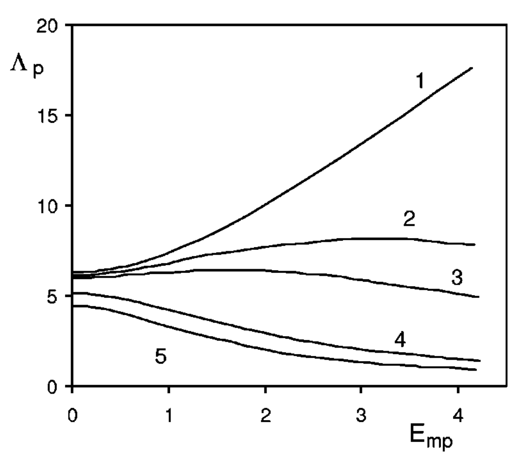


FIG. 3. Relativistic plasma wavelength $\Lambda_p$ depending on the electric field amplitude $E_{mp}$ (in dimensionless units); $\gamma=10$. $1-\mu=0$; $2-\mu=0.05$; $3-\mu=0.1$; $4-\mu=0.5$; $5-\mu=1$.

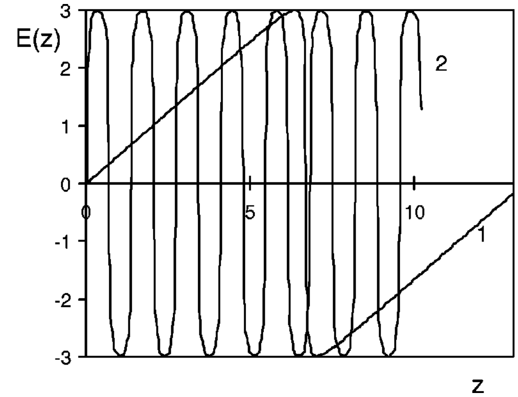


FIG. 4. Relativistic strong wave in a plasma with immobile ions ($1-\mu=0$) and in an electron-positron plasma ($2-\mu=1$); $\gamma=10$. The electric field strength $E$ and coordinate $z$ are in the normalized units.

coincides with that previously obtained [7]; in this case $\Lambda_p$ grows with amplitude due to relativistic velocities of the oscillating plasma particles (notice that in the nonlinear nonrelativistic regime the plasma wavelength does not depend on amplitude). The motion of positive ions for fixed amplitude causes the decrease of charge separation length and, therefore, leads to the decrease of the wavelength. Figure 3 clearly shows the competition between two tendencies. For the small $\mu$ the wavelength grows with amplitude due to nonlinearity. With the increase of $E_{mp}$, the effect of ion motion becomes more essential. When $\mu$ is not small, the behavior of the wavelength is caused mainly by ion motion. In electron-positron plasma ($\mu=1$), $\Lambda_p$ monotonically decreases with the increase of $E_{mp}$, in contrast to the case of heavy ions ($\mu\approx 0$). The results of simulations presented in Fig. 3 conform with the well known result of linear theory ($E_{mp}\ll 1$; see, e.g., Ref. [15]): $\Lambda_p=\Lambda_{p0}/(1+\mu)^{1/2}$, where $\Lambda_{p0}=\Lambda_p(\mu=0)$. Note also that the results practically did not change for arbitrary $\gamma\gg 1$. A considerable decrease of the relativistic plasma wavelength with $\mu$ is demonstrated in Fig. 4.

Previous studies have shown that energy of electrons (or positrons) accelerated in the field of a relativistic nonlinear wave, in a cold plasma with immobile ions, can reach a value of $4m_ec^2\gamma^3$ [7,16]. In the general case the relativistic factor of a resonant electron passing from a point with the dimensionless potential $\Phi_1$ to a point with $\Phi_2$ is equal to [7]

$$\gamma_{acc}\approx\gamma_{acc}(0)+2\gamma^2(\Phi_2-\Phi_1), \tag{12}$$

where $\gamma_{acc}(0)\approx\gamma$ is the value of the relativistic factor at the initial point. When $\mu=0$, $\Phi_{\min}=1/\gamma\leq\Phi\leq\Phi_{\max}\approx 2\gamma$ [7]. Substituting this maximum and minimum values in Eq. (12), for the maximum energy of accelerated electrons one obtains $(\gamma_{acc})_{\max}\approx 4\gamma^3$. With $\mu$ growth, the maximum energy decreases due to the decrease of $\Phi_{\max}$ (see Fig. 1). For the case of electron-positron plasma function $U(\Phi)$ is symmetric with reference to axis $\Phi=1$ and, as it is easy to see, in this case $\Phi_{\min}=1/\gamma$, $\Phi_{\max}=2-1/\gamma$. Then, the maximum energy of accelerated particles is $(\gamma_{acc})_{\max}\approx 4\gamma^2$, that is, $\gamma$ times less than that in the case $\mu=0$.

If a bunch of charged particles with density $n_b(z)$ and electric charge $q_b$ passes through a plasma, adding to the left side of Eq. (7) the value $\alpha(z)=\beta^2(q_b/|q_-|)n_b(z)/n_{0-}$, we obtain the equation that describes the excitation of steady plasma wake fields by the bunch. In this case the phase velocity is equal to the velocity of the bunch. In this section the properties of RSW's have been investigated by simulation of wake wave generation by charged bunches.

Above we have considered the case $\mu \ll 1$. However, one can see that the obtained results are valid also for $\mu > 1$, if we replace $E$ by $-E$ [this new $E$ is normalized to $m_+ \omega_p v_{ph}/q_+$, $\omega_p = (4\pi n_{0+} q_+^2/m_+)^{1/2}$], replace $\mu$ by $1/\mu$, replace the subscript $+$ by $-$, and vice versa.

### III. INFLUENCE OF ELECTRON TEMPERATURE ON RELATIVISTIC NONLINEAR PLASMA WAVES: HYDRODYNAMIC APPROACH

Here we continue to consider the dispersion properties of RSW's in the framework of the hydrodynamic approach, and investigate relativistic nonlinear waves in a warm plasma. Adding the relativistic pressure term $-(\gamma_e^2/N_e)(1-\beta\beta_e)dP/dz$ [11,14] with $P = \tau(N_e/\gamma_e)^3$ [11] (which is a relativistic generalization of the usual equation of state for one-dimensional adiabatic compression) to the equation of motion of plasma electrons, one can obtain the equations

$$(\beta-\beta_e)\frac{d(\beta_e\gamma_e)}{dz} = \beta^2 E + 3\beta^2\tau\frac{\gamma_e(1-\beta\beta_e)^2}{(\beta-\beta_e)^3}\frac{d\beta_e}{dz},$$
$$\frac{dE}{dz} = -\frac{1}{\beta^2}\frac{d^2\Phi}{dz^2} = 1 - N_e. \tag{13}$$

In Eq. (13) $\beta_e = v_e/c$ and $\tau = T/m_e c^2$ are the dimensionless velocity and temperature of plasma electrons, $\gamma_e = (1-\beta_e^2)^{-1/2}$, $\Phi = 1 + |e|\varphi/m_e c^2$. The plasma ions are assumed to be immobile due to their large mass. The density of electrons $N_e = n_e/n_0$ normalized to the unperturbed value $n_0$, as usual, is obtained from the continuity equation

$$N_e = \beta/(\beta-\beta_e). \tag{14}$$

When $\tau \to 0$ equations (13) and (14) describe RSW's in a cold plasma [5,6]. Note, that the value $\tau = 1$ corresponds to a temperature of about $6\times10^9$ K. For laboratory plasmas the temperature changes in the bounds $\tau \sim 10^{-6} \div 10^{-2}$; for star plasmas $\tau \sim 10^{-5} - 1$.

The dispersion correlation can be obtained analytically for a weakly nonlinear wave, when $u = \beta_e(z)/\beta \ll 1$. In this case from equations (13) and (14) we have

$$(a_0 - a_1 u + a_2 u^2)\frac{d^2u}{dz^2} - (a_1 - 2a_2 u)\left(\frac{du}{dz}\right)^2 + u + u^2 + u^3 = 0,$$
$$a_0 = 1 - 3\tau/\beta^2,\ a_1 = 1 + 3\tau(3-2\beta^2)/\beta^2, \tag{15}$$
$$a_2 = 3\beta^2/2 - 3\tau(6 - 11\beta^2/2 + \beta^4)/\beta^2.$$

Looking for a solution of Eq. (15) as (see, e.g., Ref. [17])

$$u = \varepsilon u_1(\Psi) + \varepsilon^2 u_2(\Psi) + \varepsilon^3 u_3(\Psi) + \cdots,$$
$$d\Psi/dz = \lambda_p/\Lambda_p = \kappa_0 + \varepsilon\kappa_1 + \varepsilon^2\kappa_2 + \cdots,$$

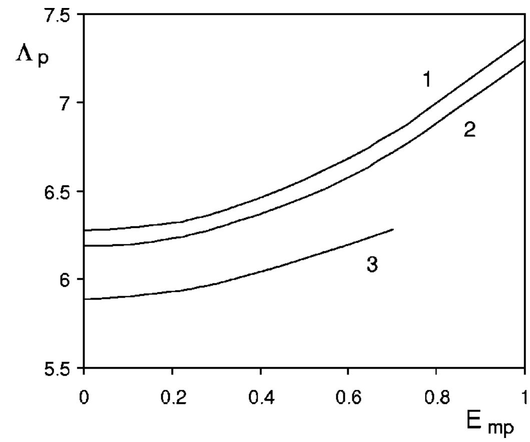


FIG. 5. Relativistic plasma wavelength in a warm plasma as a function of the wave amplitude (in the dimensionless units); $\gamma = 10$. $1-\tau = 10^{-3}$; $2-\tau = 10^{-2}$; $3-\tau = 4\times10^{-2}$.

we obtain

$$\lambda_p/\Lambda_p = a_0^{-1/2}(1 + b\beta_m^2),$$
$$b = -3/16 + 3\tau a_0^{-1}(10 - 9\beta^2 + \beta^4)/8\beta^4 \tag{16}$$
$$+ 3\tau^2 a_0^{-2}(2-\beta^2)^2/\beta^6,$$

where $\varepsilon = \beta_m/\beta \ll 1$ is the small parameter, $\beta_m = (\beta_e)_{\max}$, and $\Lambda_p$ is the wavelength. In the linear case ($\beta_m^2 \to 0$) from Eq. (16) it follows that the wavelength decreases with the temperature according to the well known Bohm-Gross dispersion correlation: $\Lambda_p = \lambda_p(1 - 3\tau/\beta^2)^{1/2}$. On the other hand, in a cold plasma ($\tau = 0$) the wavelength increases due to nonlinearity (see, e.g., Ref. [15]): $\Lambda_p \approx \lambda_p(1 + 3\beta_m^2/16)$. Figure 5 shows the dependence of the wavelength on wave amplitude in thermal plasma obtained by simulation of Eqs. (13) and (14). In the case of low temperature this dependence almost coincides with that in a cold plasma (compare curves 1 in Figs. 5 and 3).

In the case $\beta \to 1$ equations (13) and (14) can be easily integrated (see also Ref. [11]):

$$\Phi - \left(\frac{1-\beta_e}{1+\beta_e}\right)^{1/2} - 3\tau\left[\left(\frac{1+\beta_e}{1-\beta_e}\right)^{1/2} - 1\right] = 0. \tag{17}$$

One can see that the thermal term cannot be neglected near the wave breaking ($\beta_e \to 1$) even for low temperatures. In the latter case the wave-breaking field is proportional to $\tau^{-1/4}$ [11].

In the framework of hydrodynamic theory the velocity of plasma electrons cannot exceed the wave phase velocity. Actually, if $\beta_e > \beta$, then, according to expression (14), the density of electrons becomes negative, which makes no physical sense. In reality, when $\beta_e \approx \beta$ in a warm plasma (even when the temperature is low), a considerable amount of electrons get velocities more than the phase velocity due to their thermal energy distribution. When the temperature is not low, the energy distribution effect on plasma waves is essential in all cases. Therefore, the strong waves near the wave breaking and when $\tau$ is not low can be described correctly in the framework of the kinetic approach.

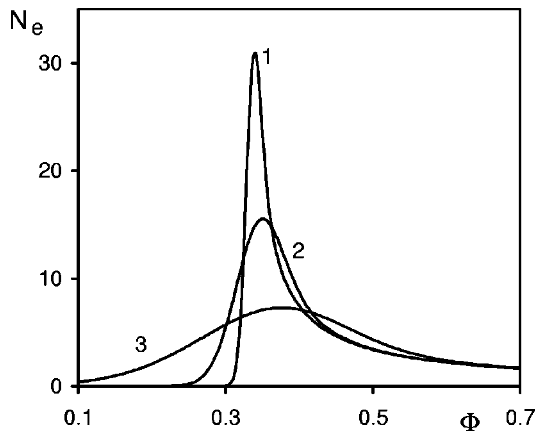


FIG. 6. Plasma electron density normalized on its equilibrium value as a function of the dimensionless electric potential $\Phi$; $\gamma=3$. $1-\tau=10^{-4}$; $2-\tau=10^{-3}$; $3-\tau=10^{-2}$.

## IV. KINETIC THEORY OF RELATIVISTIC STRONG PLASMA WAVES

As in the preceding section, here we assume plasma ions to be immobile. The kinetic approach, for one-dimensional steady fields passing through a warm plasma, gives the following system, obtained from relativistic Vlasov equation and Maxwell equations (see, e.g., Ref. [18]):

$$\left[\beta-\frac{p}{(1+p^2)^{1/2}}\right]\frac{\partial f}{\partial z}-\frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial p}=0, \tag{18}$$

$$\frac{d^2\Phi}{dz^2}+\beta^2(1-N_e)=0, \tag{19}$$

$$N_e=\int_{-\infty}^{+\infty} f(p,z)\,dp, \tag{20}$$

where $p=p_z$ is the plasma electron momentum, normalized to $m_e c$, $f(p,z)$ is the distribution function. In an unperturbed plasma the distribution function is equal to a 1D relativistic Maxwell distribution [19]

$$f_0=A[1+(1+p^2)^{1/2}/\tau]\exp[-(1+p^2)^{1/2}/\tau],$$

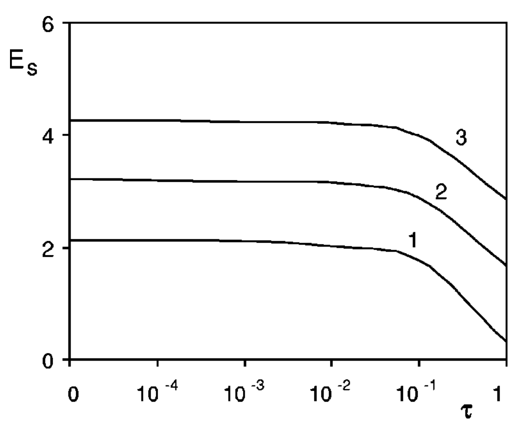


FIG. 7. Maximum value of the normalized electric field amplitude of a nonlinear plasma wave depending on the dimensionless plasma electron temperature. $1-\gamma=3$; $2-\gamma=6$; $3-\gamma=10$.

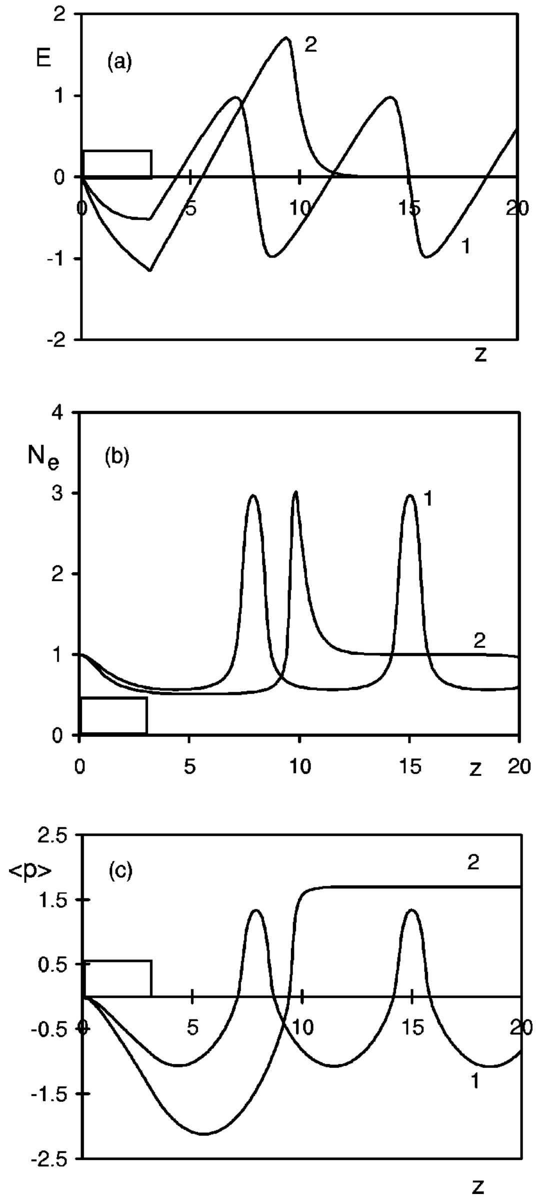


FIG. 8. Electric field strength (a), plasma electron density (b), and average electron momentum (c) in a strong plasma field, excited by a uniform electron bunch ($\gamma=3$, $\tau=0.1$). The rectangles show the bunch. 1−periodic wave, $n_b/n_0=0.4$; 2−soliton, $n_b/n_0=0.6575$. All quantities are in dimensionless units.

$$\begin{aligned} &A=\tau/2K_2(1/\tau),\\ &\langle p^2\rangle_0=\tau[K_1(1/\tau)/K_2(1/\tau)+4\tau],\\ &\langle(1+p^2)^{1/2}\rangle_0=2\tau+(1-\tau^2)K_1(1/\tau)/K_2(1/\tau), \end{aligned} \tag{21}$$

where $K_n(x)$ is the modified Bessel function of the $n$th order. In Eq. (21) we also have written out the average squared pulse and total energy for the one-dimensional equilibrium distribution $f_0$, that may be interesting for future investigations.

Solving Eq. (18) by the method of characteristics (see, e.g., Ref. [20]) and requiring that the function $f$ reduces to the equilibrium distribution (21) at $\Phi=1$, one obtains the following general solution:

$$f = A(1 + S/\tau)\exp(-S/\tau),$$

$$S = (1 + g^2)^{1/2}, \quad g = -\gamma^2[\beta r \pm (r^2 - \gamma^{-2})^{1/2}], \qquad (22)$$

$$r = \beta p - (1 + p^2)^{1/2} + \Phi - 1,$$

In the expression for $g$, the plus sign corresponds to the case $p \leqslant \beta\gamma$ and the minus sign to $p > \beta\gamma$; in equilibrium ($\Phi = 1$) we have $g = p$. Substituting the expressions (20) and (22) in Eq. (19), one obtains the equation for $\Phi$. The plasma electron density $N_e(\Phi)$ obtained numerically from Eqs. (20) and (22) is presented in Fig. 6. In the case of low temperature, the integral in Eq. (20) can be calculated by the Laplace asymptotic method [19]. The value of $N_e$ is at maximum when $\Phi \approx 1/\gamma$ and is equal to

$$N_{\max} \approx [\Gamma(1/4)/4]\gamma(\beta\gamma/\pi)^{1/2}(2/\tau)^{1/4} \approx 0.6\gamma(\beta\gamma)^{1/2}\tau^{-1/4}. \qquad (23)$$

According to expression (23), in a cold plasma $N_{\max} \rightarrow \infty$, which conforms to the previous investigations [5,6,11]. When $\tau \ll 1$ and $\Phi > 1/\gamma$, the dependence $N_e(\Phi)$ is approximately described by expression (6).

Simulations of the problem show that the plasma wavelength increases with the wave amplitude and tends to infinity for a solitary wave (soliton). Figure 7 shows dependence of the maximum value of the amplitude (which corresponds to the maximum of electric field strength in the soliton $E_s$) on plasma temperature for different $\gamma$. One can see that $E_s$ is almost constant and equal to the relativistic wave-breaking field $E_{rel}$ for values of $\tau$ up to $0.05 - 0.1$ and then decreases rapidly.

As mentioned above, the relativistic plasma waves can be excited by charged bunches or laser pulses. In order to describe the excitation of the wake field by a charged bunch, it is necessary to add in the left-hand side of Eq. (19) the value $\beta^2\alpha(z)$ [the definition of $\alpha(z)$ seen in Sec. II]. The nonlinear periodical wave and the solitary wave excited by uniform electron bunch are plotted in Fig. 8. The plasma electron density behind the soliton tends to its equilibrium value ($N_e \rightarrow 1$) and the electric field strength tends to zero. However, in this case the average plasma electron momentum $\langle p \rangle = \int_{-\infty}^{+\infty} p f(p,z) dp / \int_{-\infty}^{+\infty} f(p,z) dp$ tends to a nonzero constant value. This does not seem strange because the solitary wave can be considered as a wave with infinite wavelength. Hence, bulk motion of plasma electrons behind the solitary wave takes place, while the plasma remains neutral. Equations (18)–(21) have also nonperiodical solutions. However, such solutions have no physical sense [19] and should be considered in the context of nonstationary kinetic theory. Thus, in a thermal plasma with immobile ions two kinds of steady waves can exist: periodical waves and solitons.

## V. CONCLUSIONS

The results presented in this paper supplement the theory of nonlinear relativistic plasma waves, taking into consideration the motion of ions and finite plasma temperature. It is shown that the nonlinearity leads to the increase of relativistic wavelength, while ion motion leads to the wavelength decrease. For example, in electron-positron plasma the wavelength monotonously decreases as the amplitude increases. The relativistic wave-breaking field weakly depends on the ion mass.

Contrary to the case of cold plasma, in a warm plasma the relativistic solitary waves (solitons) can exist. The plasma wavelength grows monotonously with the amplitude in the warm plasma due to nonlinearity. It has been found that the maximum electron density in the plasma wave decreases with the temperature as $T^{-1/4}$ and tends to infinity in a cold plasma, as was shown by previous investigations.

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